Attitude Stability of Satellites Subjected to Gravity Gradient and Aerodynamic Torques

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The effect of aerodynamic and gravitational torques on the attitude stability of orbiting satellites is investigated. The satellite is assumed to be a rigid body on a circular orbit. No restrictions are made relative to mass and shape properties. Liapunov's direct method is applied in case of conservative aerodynamic torques to investigate the stability of the satellite's equilibrium orientations relative to an orbiting reference frame. At least one stable equilibrium exists if the aerodynamic torque is conservative not only in the vicinity of the equilibrium but also for arbitrary orientations of the satellite. This property is determined by the shape of the satellite and the location of the center of mass. The stability investigation by linearized analysis shows that the equilibrium orientations are in general unstable if the aerodynamic torque is nonconservative.

I. Introduction

At altitudes up to about 800 km, aerodynamic torques affecting orbiting satellites can be of the same order of magnitude as gravitational torques and, consequently, can have a considerable effect on the dynamic behavior of the satellite. But even if aerodynamic torques are much smaller than gravity-gradient torques, they can cause a destabilization of gravity-gradient stabilized satellites.

The influence of aerodynamic and gravitational torques on the attitude stability of satellites in circular orbits and in an atmosphere of constant density has been treated in several papers, but only satellites with special shape and mass properties have been considered.

Meirovitch and Wallace[1] have investigated the stability of a satellite that is spinning about an axis of dynamic and geometric symmetry. Beletskii[2] has considered the attitude stability of a body of revolution whose axis of geometric symmetry is coincident with a principal axis of inertia. In Refs. 1 and 2 stable equilibrium orientations could be found because of the shape and mass properties of the satellites under consideration. Nurre[3] has shown that arbitrarily small aerodynamic torques can cause a destabilization of gravity-gradient stabilized satellites. He has investigated a satellite whose center of mass is in a plane of geometric symmetry and whose axis of maximal moment of inertia is perpendicular to this plane.

The present investigation makes no restrictions with respect to mass distribution and shape. The satellite is considered to be an arbitrary, rigid body. A circular orbit is assumed and the atmospheric density is taken to be constant. The nonuniformity of the Earth's gravity field is ignored.

Coordinate Systems

The following coordinate systems (Fig. 1) are used.

1) The orbital coordinate system is designated 0_x, the origin of which is the satellite's center of mass. The x_3 axis is directed along the radius vector connecting the center of mass of the Earth and the satellite. The x_1 axis is tangential to the orbit in flight direction and the x_2 axis completes 0_x to a right-handed coordinate system.

2) The body-fixed coordinate system is designated 0_y. The y_1, y_2, and y_3 axes are the satellite's principal axes of inertia unless otherwise stated. The moments of inertia are I_1, I_2, I_3. The orientation of 0_y relative to 0_x is described by three successive positive rotations $\alpha_1, \alpha_2, \alpha_3$ about the axes 1, 2, 3.

The transformation matrix Θ between 0_x and 0_y defined by $\mathbf{y} = \Theta\mathbf{x}$ is with s = sin, c = cos

$$\Theta = \begin{bmatrix} c\alpha_2 c\alpha_3 & s\alpha_1 s\alpha_2 c\alpha_3 + c\alpha_1 s\alpha_3 & s\alpha_1 s\alpha_3 - c\alpha_1 s\alpha_2 c\alpha_3 \\ -c\alpha_2 s\alpha_3 & c\alpha_1 c\alpha_3 - s\alpha_1 s\alpha_2 s\alpha_3 & s\alpha_1 c\alpha_3 + c\alpha_1 s\alpha_2 s\alpha_3 \\ s\alpha_2 & -s\alpha_1 c\alpha_2 & c\alpha_1 c\alpha_2 \end{bmatrix} \tag{1}$$

3) The orientations of the coordinate systems 0_w and 0_z with respect to 0_x are determined by the rotations α_1 and α_1,α_2, respectively.

Gravity Gradient Torque

The gravity gradient torque described by its components in the principal axes system is

$$\mathbf{M}_G = 3\Omega_0^2 \begin{bmatrix} (I_3 - I_2)\Theta_{33}\Theta_{23} \\ (I_1 - I_3)\Theta_{13}\Theta_{33} \\ (I_2 - I_1)\Theta_{23}\Theta_{13} \end{bmatrix} \tag{2}$$

where Θ_{i3} ($i = 1,2,3$) are the elements of the matrix Θ, Eq. (1), and Ω_0 is the orbital angular velocity.

The gravity gradient torque is a conservative torque, the potential of which is

$$U_G = \tfrac{3}{2}\Omega_0^2(I_1\Theta_{13}^2 + I_2\Theta_{23}^2 + I_3\Theta_{33}^2) + C_1 \tag{3}$$

where C_1 is an arbitrary constant.

Aerodynamic Torque

The aerodynamic torque affecting the satellite depends upon the orientation of a body-fixed geometric coordinate system with respect to the flow vector. This orientation is described by two angles. Since the aerodynamic torque is a periodic function of the two angles, it can be approximated by a finite number of terms of a Fourier series.[3] This fact is valid regardless of what mechanism is assumed for the interaction between the gas molecules and the satellite's surface.

We can assume with sufficient accuracy that the oncoming stream of molecules is in the negative x_1 direction, which

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implies that the rotation of the atmosphere due to the earth's rotation is neglected. For the sake of simplicity we can further assume, without loss in generality, that the geometric coordinate system is aligned with the principal axes system. Then the orientation of the geometric system with respect to the flow vector is described by the angles α_2 and α_3.

In the following discussion we will assume that the aerodynamic torque is a known periodic function of α_2 and α_3. It will be denoted by $\mathbf{M}_{Ax}$, $\mathbf{M}_{Ay}$, $\mathbf{M}_{Aw}$, or $\mathbf{M}_{Az}$ if it is given in 0_x, 0_y, 0_w, or 0_z.

The aerodynamic torque is conservative if the work performed on any closed path is equal to zero

$$W = \oint (M_{Ax_1}d\alpha_1 + M_{Aw_2}d\alpha_2 + M_{Ay_3}d\alpha_3) = 0 \tag{4}$$

From condition (4) follows[4] that the curl of $\mathbf{M}_A$ must vanish identically. Hence the conditions for conservative aerodynamic torque are

$$(\partial M_{Aw_2}/\partial\alpha_1) - (\partial M_{Ax_1}/\partial\alpha_2) = 0 \tag{5}$$

$$(\partial M_{Ay_3}/\partial\alpha_2) - (\partial M_{Aw_2}/\partial\alpha_3) = 0 \tag{6}$$

$$(\partial M_{Ax_1}/\partial\alpha_3) - (\partial M_{Ay_3}/\partial\alpha_1) = 0 \tag{7}$$

As mentioned previously, the aerodynamic torque does not depend on α_1. With Eqs. (5) and (7) we get therefore

$$\partial M_{Ax_1}/\partial\alpha_i = 0,\ i = 1,2,3 \tag{8}$$

and the conditions for the aerodynamic torque to be conservative become

$$M_{Ax_1} = \text{const} \tag{9}$$

$$\partial M_{Ay_3}/\partial\alpha_2 = \partial M_{Aw_2}/\partial\alpha_3 \tag{10}$$

A shape of the satellite for which the aerodynamic torque about the x_1 axis is independent of the satellite's orientation and not zero, is hardly conceivable; therefore, M_{Ax_1} will in general be zero in case of conservative aerodynamic torque.

Conditions (9) and (10) are satisfied for all bodies of revolution if the center of mass is on the axis of geometric symmetry. Let for the moment 0_y be a geometric coordinate system with y_3 being the axis of geometric symmetry and not necessarily a principal axis of inertia. Then the x_1,y_3 plane is always a plane of geometric symmetry, and the aerodynamic torque has no component in the x_1 direction. Furthermore, M_{Ay_3} is identically zero and M_{Aw_2} does not depend on α_3. Consequently, the aerodynamic torque is conservative.

However, the aerodynamic torque is nonconservative if the center of mass is not on the axis of symmetry. In this case, condition (9) is violated. The aerodynamic torque is also conservative if the center of pressure is body-fixed and the aerodynamic force is constant and directed opposite to the flight direction. This holds for bodies composed of an arbitrary number of spheres (e.g., a dumbbell-shaped satellite) if no shadowing takes place and if the aerodynamic force on the connecting rods is neglected. It may also be a good approximation for other shapes if only small deviations from the equilibrium position are considered.

For conservative aerodynamic torque, a potential function U_A can be derived. The potential is equal to the negative work performed by the aerodynamic torque if the satellite is moved from a reference orientation α_{i_r} to the orientation α_i $(i = 1,2,3)$

$$U_A = -M_{Ax_1}(\alpha_1 - \alpha_{1r}) - \int_{\alpha_{2r}}^{\alpha_2} M_{Aw_2}(\eta,\alpha_{3r})d\eta - \int_{\alpha_{3r}}^{\alpha_3} M_{Ay_3}(\alpha_2,\eta)d\eta + C_2 \tag{11}$$

where C_2 is an arbitrary constant. A suitable reference position could be $\alpha_{ir} = 0$ or the equilibrium position α_{io}, the stability of which is to be investigated.

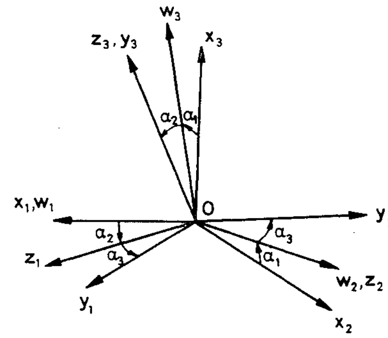

Fig. 1 Coordinate systems.

For a body of revolution whose axis of geometric symmetry is coincident with the y_3 or the y_2 axis, the aerodynamic potential will be only a function of α_2 or α_3, respectively.

II. Stability Investigation in Case of Conservative Aerodynamic Torque

According to Liapunov's direct method, an equilibrium position E is shown to be stable if a function can be found for the system which is positive-definite in the neighborhood of E, and the total derivative of this function with respect to time is negative semidefinite for the perturbed motion.[5] Such a function, if it exists, is called a Liapunov function for the system.

For conservative systems, the Hamiltonian function H may in certain cases be a suitable Liapunov function. The Hamiltonian is constant for the perturbed motion and, therefore, the total derivative dH/dt is identically zero. Hence the equilibrium position is stable if H turns out to be positive-definite in the neighborhood of the equilibrium.

Hamiltonian Function

The Hamiltonian is defined as

$$H = \sum_{i=1}^{3} \frac{\partial L}{\partial \dot{\alpha}_i}\dot{\alpha}_i - L \tag{12}$$

where

$$L = T - U \tag{13}$$

is the Lagrangian function and T and U are the kinetic and potential energy, respectively.

The kinetic energy is

$$T = \tfrac{1}{2}(I_1\omega_{y_1}{}^2 + I_2\omega_{y_2}{}^2 + I_3\omega_{y_3}{}^2) \tag{14}$$

The angular velocity $\boldsymbol{\omega}_y$ of the satellite is

$$\boldsymbol{\omega}_y = \begin{bmatrix} \dot{\alpha}_1 c\alpha_2 c\alpha_3 + \dot{\alpha}_2 s\alpha_3 + \Omega_0(c\alpha_1 s\alpha_3 + s\alpha_1 s\alpha_2 c\alpha_3) \\ -\dot{\alpha}_1 s\alpha_3 c\alpha_2 + \dot{\alpha}_2 c\alpha_3 + \Omega_0(c\alpha_1 c\alpha_3 - s\alpha_1 s\alpha_2 s\alpha_3) \\ \dot{\alpha}_1 s\alpha_2 + \dot{\alpha}_3 - \Omega_0 s\alpha_1 c\alpha_2 \end{bmatrix} \tag{15}$$

The kinetic energy T can be written in the form

$$T = T_0 + T_1 + T_2 \tag{16}$$

where T_n $(n = 0,1,2)$ is a homogeneous form of nth degree in the $\dot{\alpha}_i$.

The potential U is composed of the aerodynamic potential U_A, Eq. (11), and the gravitational potential U_G, Eq. (3). With Eqs. (16) and (13), the Hamiltonian becomes[6]

$$H = T_2 + V \tag{17}$$

V is the dynamic potential

$$V = U_G + U_A - T_0 \tag{18}$$

Equilibrium Conditions

The equilibrium orientations E correspond to stationary values of the dynamic potential. Therefore, the conditions

$$V_i \equiv (\partial V/\partial \alpha_i)|_{\text{at } E} = 0, \; i = 1,2,3 \tag{19}$$

must be satisfied. If α_{10}, α_{20}, and α_{30} denote an equilibrium orientation E, the derivatives V_i are

$$\begin{aligned} V_1 &= 4\Omega_0^2\{(I_2 - I_1)\mathrm{s}\alpha_{20}\mathrm{s}\alpha_{30}\mathrm{c}\alpha_{30}(\mathrm{c}^2\alpha_{10} - \mathrm{s}^2\alpha_{10}) + \\ &\quad \mathrm{s}\alpha_{10}\mathrm{c}\alpha_{10}[(I_2 - I_1)(\mathrm{c}^2\alpha_{30} - \mathrm{s}^2\alpha_{20}\mathrm{s}^2\alpha_{30}) + \\ &\quad (I_1 - I_3)\mathrm{c}^2\alpha_{20}]\} + U_{A_1} \\ V_2 &= 4\Omega_0^2(I_2 - I_1)\mathrm{s}\alpha_{10}\mathrm{c}\alpha_{10}\mathrm{c}\alpha_{20}\mathrm{s}\alpha_{30}\mathrm{c}\alpha_{30} + \\ &\quad \Omega_0^2\mathrm{s}\alpha_{20}\mathrm{c}\alpha_{20}(3\mathrm{c}^2\alpha_{10} - \mathrm{s}^2\alpha_{10}) \times \\ &\quad \{(I_2 - I_1)\mathrm{s}^2\alpha_{30} + I_1 - I_3\} + U_{A_2} \\ V_3 &= \Omega_0^2(I_2 - I_1)\{4\mathrm{s}\alpha_{10}\mathrm{c}\alpha_{10}\mathrm{s}\alpha_{20}(\mathrm{c}^2\alpha_{30} - \mathrm{s}^2\alpha_{30}) + \\ &\quad \mathrm{s}\alpha_{30}\mathrm{c}\alpha_{30}(4\mathrm{s}^2\alpha_{20}\mathrm{c}^2\alpha_{10} - 4\mathrm{s}^2\alpha_{10} + \mathrm{c}^2\alpha_{20})\} + U_{A_3} \end{aligned} \tag{20}$$

with $U_{Ai} = (\partial U_A/\partial \alpha_i)|_{\text{at } E}$

Stability Conditions

As mentioned previously, a sufficient condition for E to be stable is that the Hamiltonian is positive-definite in the neighborhood of E.†

In determining the positive definiteness of H, the term T_2 can be ignored since it is always positive-definite. The stability condition is then as follows. The equilibrium position is Liapunov-stable if the dynamic potential V is positive-definite in the neighborhood of E. An equivalent condition is that V must have a relative minimum at E. If V has a relative minimum, one can make V positive-definite by suitably choosing C_1 in Eq. (3).

The necessary and sufficient conditions for a relative minimum of V at E are[7]

$$V_{11} > 0; \quad \begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} > 0; \quad \begin{vmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{vmatrix} > 0 \tag{21}$$

$$V_{ij} \equiv (\partial^2 V/\partial\alpha_i\partial\alpha_j)|_{\text{at } E} \; (i,j = 1,2,3)$$

Computing the second partial derivatives at E, we get

$$\begin{aligned} V_{11} &= -16\Omega_0^2(I_2 - I_1)\mathrm{s}\alpha_{10}\mathrm{c}\alpha_{10}\mathrm{s}\alpha_{20}\mathrm{s}\alpha_{30}\mathrm{c}\alpha_{30} + \\ &\quad 4\Omega_0^2(\mathrm{c}^2\alpha_{10} - \mathrm{s}^2\alpha_{10})\{(I_1 - I_3)\mathrm{c}^2\alpha_{20} + \\ &\quad (I_2 - I_1)(\mathrm{c}^2\alpha_{30} - \mathrm{s}^2\alpha_{20}\mathrm{s}^2\alpha_{30})\} \\ V_{12} &= 4\Omega_0^2(I_2 - I_1)\mathrm{c}\alpha_{20}\mathrm{s}\alpha_{30}\mathrm{c}\alpha_{30}(\mathrm{c}^2\alpha_{10} - \mathrm{s}^2\alpha_{10}) - \\ &\quad 8\Omega_0^2\mathrm{s}\alpha_{10}\mathrm{c}\alpha_{10}\mathrm{s}\alpha_{20}\mathrm{c}\alpha_{20}\{(I_2 - I_1)\mathrm{s}^2\alpha_{30} + I_1 - I_3\} \\ V_{22} &= -4\Omega_0^2(I_2 - I_1)\mathrm{c}\alpha_{10}\mathrm{s}\alpha_{10}\mathrm{s}\alpha_{20}\mathrm{s}\alpha_{30}\mathrm{c}\alpha_{30} + \\ &\quad \Omega_0^2(3\mathrm{c}^2\alpha_{10} - \mathrm{s}^2\alpha_{10})(\mathrm{c}^2\alpha_{20} - \mathrm{s}^2\alpha_{20}) \times \\ &\quad \{(I_2 - I_1)\mathrm{s}^2\alpha_{30} + I_1 - I_3\} + U_{A_{22}} \\ V_{13} &= -4\Omega_0^2(I_2 - I_1)\{2\mathrm{s}\alpha_{10}\mathrm{c}\alpha_{10}\mathrm{s}\alpha_{30}\mathrm{c}\alpha_{30}(1 + \mathrm{s}^2\alpha_{20}) + \\ &\quad \mathrm{s}\alpha_{20}(\mathrm{s}^2\alpha_{10} - \mathrm{c}^2\alpha_{10})(\mathrm{c}^2\alpha_{30} - \mathrm{s}^2\alpha_{30})\} \\ V_{23} &= 2\Omega_0^2\mathrm{c}\alpha_{20}(I_2 - I_1)\{2\mathrm{s}\alpha_{10}\mathrm{c}\alpha_{10}(\mathrm{c}^2\alpha_{30} - \mathrm{s}^2\alpha_{30}) - \\ &\quad \mathrm{s}\alpha_{20}\mathrm{s}\alpha_{30}\mathrm{c}\alpha_{30}(\mathrm{s}^2\alpha_{10} - 3\mathrm{c}^2\alpha_{10})\} + U_{A_{23}} \\ V_{33} &= \Omega_0^2(I_2 - I_1)\{-16\mathrm{s}\alpha_{10}\mathrm{c}\alpha_{10}\mathrm{s}\alpha_{20}\mathrm{s}\alpha_{30}\mathrm{c}\alpha_{30} + \\ &\quad (4\mathrm{s}^2\alpha_{20}\mathrm{c}^2\alpha_{10} - 4\mathrm{s}^2\alpha_{10} + \mathrm{c}^2\alpha_{20}) \times \\ &\quad (\mathrm{c}^2\alpha_{30} - \mathrm{s}^2\alpha_{30})\} + U_{A_{33}} \\ U_{Aij} &\equiv (\partial^2 U_A/\partial\alpha_i\partial\alpha_j)|_{\text{at } E}, \; V_{ij} = V_{ji} \end{aligned} \tag{22}$$

† For a satellite whose y_3-axis is an axis of geometric as well as dynamic symmetry $\{I_1 = I_2$ and $U_A(\alpha_2)\}$, the Hamiltonian does not depend on α_3 and, therefore, cannot be positive-definite. However, in this case, α_3 is a cyclic coordinate and can be replaced by a first integral of the motion. This special case is treated in Ref. 1 and will be excluded from the further discussion.

Example 1

If the center of pressure is body-fixed and the aerodynamic force is constant and directed against the flight direction, the aerodynamic potential is

$$U_A = |\mathbf{F}_A|x_{CP1} = |\mathbf{F}_A|(y_{CP1}\mathrm{c}\alpha_2\mathrm{c}\alpha_3 - y_{CP2}\mathrm{c}\alpha_2\mathrm{s}\alpha_3 + y_{CP3}\mathrm{s}\alpha_2) \tag{23}$$

where x_{CPi} and y_{CPi} are the coordinates of the center of pressure in 0_x and 0_y, respectively. $\mathbf{F}_A$ is the aerodynamic force.

Case I: Assuming an equilibrium for which two principal axes of inertia are in the orbital plane, the equilibrium orientation can be described by $\alpha_{10} = \alpha_{30} = 0$ and $0 \leqq \alpha_{20} < \pi/2$. α_{20} can be determined from Eq. (19) with Eqs. (20) and (23)

$$\begin{aligned} 3\Omega_0^2\mathrm{s}\alpha_{20}\mathrm{c}\alpha_{20}(I_1 - I_3) &= |\mathbf{F}_A|(y_{CP1}\mathrm{s}\alpha_{20} - y_{CP3}\mathrm{c}\alpha_{20}) \\ y_{CP2} &= 0 \end{aligned} \tag{24}$$

In this case the center of pressure must be in the orbital plane. Using Eq. (24) the stability conditions (21) become

$$(I_2 - I_1) + (I_1 - I_3)\mathrm{c}^2\alpha_{20} > 0$$

$$y_{CP1} < \frac{\Omega_0^2\mathrm{c}\alpha_{20}(I_2 - I_1)\{I_2 - I_1 + (I_1 - I_3)(1 + 3\mathrm{s}^2\alpha_{20})\}}{|\mathbf{F}_A|\{I_2 - I_1 + (I_1 - I_3)\mathrm{c}^2\alpha_{20}\}} \tag{25}$$

$$y_{CP1} < \frac{3\Omega_0^2\mathrm{c}^3\alpha_{20}(I_1 - I_3)}{|\mathbf{F}_A|}$$

Case II: For an equilibrium with two principal axes of inertia being in the x_1,x_2 plane, the orientation can be described by $\alpha_{10} = \alpha_{20} = 0$ and $0 \leqq \alpha_{30} \leqq \pi/2$. The equilibrium conditions are

$$\begin{aligned} \Omega_0^2(I_2 - I_1)\mathrm{s}\alpha_{30}\mathrm{c}\alpha_{30} &= |\mathbf{F}_A|(y_{CP1}\mathrm{s}\alpha_{30} + y_{CP2}\mathrm{c}\alpha_{30}) \\ y_{CP3} &= 0 \end{aligned} \tag{26}$$

In the equilibrium orientation the center of pressure must be in the x_1,x_2 plane. The stability conditions are

$$\begin{aligned} &(I_2 - I_1)\mathrm{c}^2\alpha_{30} + I_1 - I_3 > 0 \\ &y_{CP1} < \Omega_0^2(I_2 - I_1)\mathrm{c}^3\alpha_{30}/|\mathbf{F}_A| \\ &y_{CP1} < \frac{\Omega_0^2\mathrm{c}\alpha_{30}(I_1 - I_3)\{3(I_2 - I_3) + (I_2 - I_1)\mathrm{s}^2\alpha_{30}\}}{|\mathbf{F}_A|\{(I_2 - I_1)\mathrm{c}^2\alpha_{30} + I_1 - I_3\}} \end{aligned} \tag{27}$$

In addition to cases I and II there are equilibrium orientations possible wherein none of the x_i,x_j planes are two principal axes of inertia. This case will be treated in example 3.

Determination of Stable Equilibrium Orientations by Gradient Methods

Instead of determining stable equilibrium orientations as described previously, it is sometimes easier to determine the minima of V, Eq. (18), numerically by gradient methods, e.g., the steepest descent method. The values α_{10}, α_{20}, α_{30} which minimize V describe the stable equilibrium orientation of the satellite.

Because V is in general a continuous, periodic function of α_1, α_2, α_3, with finite values, it has one absolute minimum and possibly one or several relative minima. Therefore, at least one stable equilibrium orientation exists.

Example 2

The procedure followed in this example can be applied to any body of revolution with the center of mass being on the axis of symmetry. A cylindrical body of length L and diameter D (Fig. 2) is considered. We want to determine the stable equilibrium orientations for different orientations of the axis of geometric symmetry with respect to the principal axes system.

The moments of inertia are assumed to be $I_2 > I_1 > I_3$. Without aerodynamic torque the stable equilibrium orientation would be $\alpha_{10} = \alpha_{20} = \alpha_{30} = 0$. In this example it is advantageous to introduce a geometric coordinate system which is not aligned with the principal axes system, to avoid a recalculation of the aerodynamic potential in each considered case. The origin 0′ of the geometric coordinate system $0_g'$ is the center of the cylinder, the g_3 axis is the cylinder's axis, and the g_1 axis is in the x_1,g_3 plane. The body's center of mass 0 is on the g_3 axis.

The aerodynamic force $\mathbf{F}_A$ is always in the g_1,g_3 plane and can be expressed by two force coefficients: the normal force coefficient c_{FN} and the axial force coefficient c_{FA}

$$\mathbf{F}_{Ag} = K\begin{bmatrix} c_{FN} \\ 0 \\ c_{FA} \end{bmatrix} \tag{28}$$

where

$$K = \tfrac{1}{2}\rho v^2 A_{\mathrm{Ref}} \tag{29}$$

with ρ being the atmospheric density, v the relative velocity of the oncoming stream, and A_{Ref} the reference area for the satellite. The aerodynamic torque about 0 is

$$\mathbf{M}_{Ag} = \begin{bmatrix} 0 \\ g_{CM3}c_{FN}K \\ 0 \end{bmatrix} \tag{30}$$

where g_{CM3} is the g_3 coordinate of the center of mass. The normal coefficient c_{FN} is an odd function of the angle of attack β between the g_3 axis and the x_1 axis

$$c_{FN} = \Sigma_i b_i si\beta \tag{31}$$

where

$$b_1 = 0.860 + 2.346L/D,\ b_3 = -0.499 + 0.420L/D$$
$$b_5 = -0.111 - 0.055L/D,\ b_7 = -0.047 + 0.016L/D$$
$$b_9 = -0.024 - 0.006L/D,\ b_{2_n} = 0 \qquad n = 0,1,2,3\ldots$$

These values were obtained from data furnished by the Lockheed Missiles and Space Co., Huntsville Research and Engineering Center, Huntsville, Ala.

The aerodynamic potential is

$$U_A = \int M_{Ag2}d\beta$$

or

$$U_A = -g_{CM3}K\Sigma_i b_i ci\beta/i + C_2 \tag{32}$$

U_A is an even function of the angle of attack. This holds for any body of revolution.

The orientation of the g_3 axis with respect to the principal axes system can be expressed by the direction cosines κ_1, κ_2, κ_3 between the positive g_3 axis and the y_1, y_2, y_3 axis, respectively. It is

$$c\beta = \kappa_1 c\alpha_2 c\alpha_3 - \kappa_2 c\alpha_2 s\alpha_3 + \kappa_3 s\alpha_2 \tag{33}$$

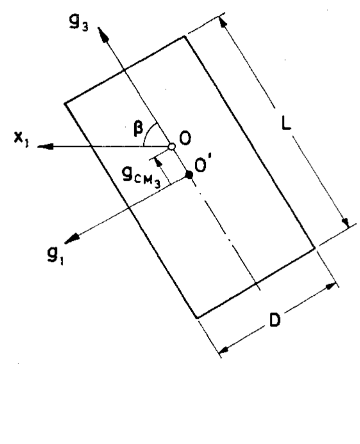

Fig. 2 Cylindrical satellite.

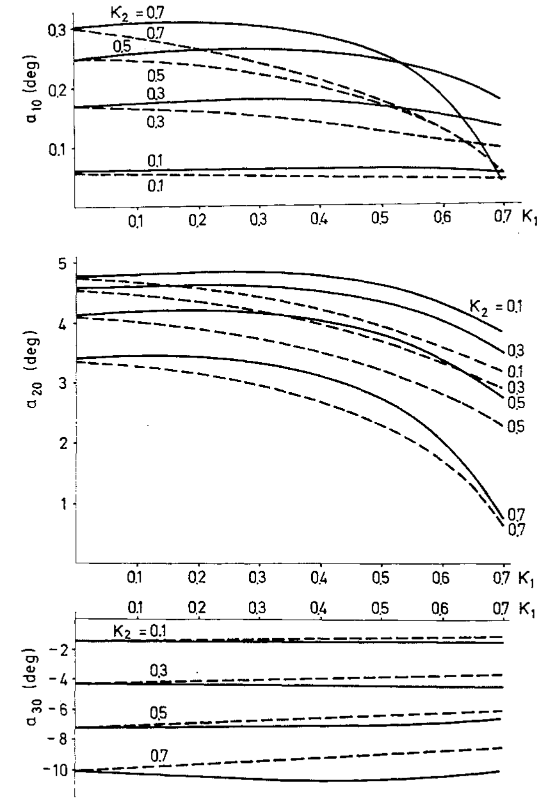

Fig. 3 Stable equilibrium orientations: example 2, case I (——); example 3 (---).

and

$$\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 1 \tag{34}$$

To normalize the function V, Eq. (18), we divide by $(I_2 - I_1)\Omega_0^2/2$ and get

$$V' = 3(s\alpha_1 c\alpha_3 + c\alpha_1 s\alpha_2 s\alpha_3)^2 - (c\alpha_1 c\alpha_3 - s\alpha_1 s\alpha_2 s\alpha_3)^2 + (s^2\alpha_1 - 3c^2\alpha_1)\eta c^2\alpha_2 - \epsilon\Sigma_i b_i ci\beta/i + C_3 \tag{35}$$

with

$$V' = 2V/[(I_2 - I_1)\Omega_0^2],\ \eta = (I_1 - I_3)/(I_2 - I_1) \tag{36}$$

and

$$\epsilon = 2Kg_{CM3}/[(I_2 - I_1)\Omega_0^2] \tag{37}$$

C_3 is an arbitrary constant.

For given values of η, ϵ, L/D, κ_1, κ_2, the stable equilibrium orientations can be determined by seeking the angles α_{i_0} ($i = 1,2,3$) which minimize V'.

Figures 3–5 show, for $L/D = 2$, α_{i_0} as functions of κ_1 and κ_2, i.e., as functions of the orientation of the axis of geometric symmetry with respect to the principal axes. For case I: $\eta = 1$; $\epsilon = 0.1$ (Fig. 3), for case II: $\eta = 0.1$; $\epsilon = 0.01$ (Fig. 4), and for case III: $\eta = 10$; $\epsilon = 1$ (Fig. 5). The values for α_{20} are almost identical in all three cases and have, therefore, been plotted only for case I.

Example 3

We will return to the problem of a satellite with a body-fixed center of pressure and with the constant aerodynamic force directed opposite to the flight direction. Let the moments of inertia be $I_2 > I_1 > I_3$.

Introducing the distance r between the center of mass and the center of pressure, the normalized function V' can be written in form of Eq. (35) with V' and η being the abbreviations, Eq. (36), and

$$\epsilon = -2r|\mathbf{F}_A|/\Omega_0^2(I_2 - I_1),\ \kappa_i = y_{CPi}/r$$
$$b_1 = 1,\ bj = 0,\ j \neq \tag{38}$$

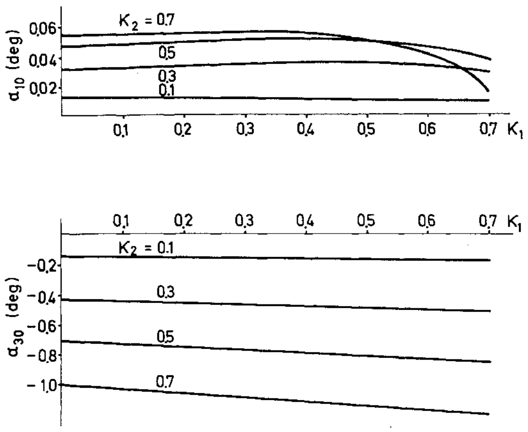

Fig. 4 Stable equilibrium orientations: example 2, case II.

For $\epsilon = 0$, i.e., for zero aerodynamic torque, the stable orientation is again $\alpha_{10} = \alpha_{20} = \alpha_{30} = 0$. The broken lines in Fig. 3 show the stable orientations for $\eta = 1$ and $\epsilon = 0.5$ as functions of κ_1 and κ_2.

III. Stability Investigation in Case of Arbitrary Aerodynamic Torque

In the following discussion the assumption of conservative aerodynamic torque will be dropped. Because the Hamiltonian is then no longer a suitable Liapunov function, stability will be investigated by linearized analysis.

Equations of Motion

The motion of the satellite can be described either by the Euler equations or by the Lagrange equations being

$$
\begin{aligned}
(d/dt)[\partial L/\partial \dot{\alpha}_1] - (\partial L/\partial \alpha_1) &= M_{Ax_1} \\
(d/dt)[\partial L/\partial \dot{\alpha}_2] - (\partial L/\partial \alpha_2) &= M_{Aw_2} \\
(d/dt)[\partial L/\partial \dot{\alpha}_3] - (\partial L/\partial \alpha_3) &= M_{Ay_3}
\end{aligned}
\tag{39}
$$

with $L = T - U_G$, where T and U_G are expressions (14) and (3), respectively.

In this paper, the Lagrange equations are used because of certain symmetry properties of the linearized equations. After inserting Eqs. (3) and (14) into Eq. (39) and setting

$$\ddot{\alpha}_i = \dot{\alpha}_i, = 0, \; i = 1,2,3$$

we get the equilibrium conditions

$$V_1 = M_{Ax_1},\; V_2 = M_{Aw_2},\; V_3 = M_{Ay_3} \tag{40}$$

where V_i are the expressions (20) with $U_{Ai} \equiv 0$. To investigate the stability of an equilibrium orientation E, the equations of motion are linearized in the vicinity of E introducing the new variables δ_i

$$\delta_i = \alpha_i - \alpha_{io}, \; i = 1,2,3 \tag{41}$$

The δ_i's are small angles.

If we neglect all small terms of second and higher order, we get linearized differential equations which can be written in the form

$$A\ddot{\boldsymbol{\delta}} + G\dot{\boldsymbol{\delta}} + C\boldsymbol{\delta} = 0 \tag{42}$$

with

$$\boldsymbol{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

A is a symmetric matrix with the elements

$$
\begin{gathered}
A_{11} = c^2\alpha_{20}(I_1c^2\alpha_{30} + I_2s^2\alpha_{30}) + I_3s^2\alpha_{20} \\
A_{12} = (I_1 - I_2)c\alpha_{20}s\alpha_{30}c\alpha_{30} \\
A_{13} = I_3s\alpha_{20},\; A_{22} = I_1s^2\alpha_{30} + I_2c^2\alpha_{30} \\
A_{23} = 0,\; A_{33} = I_3
\end{gathered}
\tag{43}
$$

The elements of the skewsymmetric matrix G are

$$
\begin{aligned}
G_{12} = {} & \Omega_0(I_2 - I_1)s\alpha_{20}\{2c\alpha_{10}s\alpha_{30}c\alpha_{30} + s\alpha_{10}s\alpha_{20}(c^2\alpha_{30} - s^2\alpha_{30})\} + (I_1 + I_2)\Omega_0c^2\alpha_{20}s\alpha_{10} - \\
& I_3\Omega_0s\alpha_{10}(c^2\alpha_{20} - s^2\alpha_{20}) \\
G_{13} = {} & \Omega_0(I_1 - I_2)c\alpha_{20}\{c\alpha_{10}(c^2\alpha_{30} - s^2\alpha_{30}) - 2s\alpha_{10}s\alpha_{20}c\alpha_{30}s\alpha_{30}\} + I_3\Omega_0c\alpha_{10}c\alpha_{20} \\
G_{23} = {} & \Omega_0(I_1 - I_2)\{s\alpha_{10}s\alpha_{20}(c^2\alpha_{30} - s^2\alpha_{30}) + 2s\alpha_{30}c\alpha_{30}c\alpha_{10}\} - I_3\Omega_0s\alpha_{10}s\alpha_{20}
\end{aligned}
\tag{44}
$$

and the elements of C are

$$
\begin{aligned}
C_{1j} &= -(\partial M_{Ax_1}/\partial \alpha_j)|_{\text{at } E} + V_{1j} \\
C_{2j} &= -(\partial M_{Aw_2}/\partial \alpha_j)|_{\text{at } E} + V_{2j} \\
C_{3j} &= -(\partial M_{Ay_3}/\partial \alpha_j)|_{\text{at } E} + V_{3j},\; j = 1,2,3
\end{aligned}
\tag{45}
$$

where the V_{ij} are the expressions (22) with $U_{Aij} \equiv 0$. C is an unsymmetric matrix unless the aerodynamic torque satisfies conditions (5–7) for conservative torques at the equilibrium position.

Stability Conditions

The characteristic equation of (42) is

$$|A\lambda^2 + G\lambda + C| = 0 \tag{46}$$

Equation (46) is of sixth degree in λ and can be written as

$$d_6\lambda^6 + d_5\lambda^5 + d_4\lambda^4 + d_3\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0 = 0 \tag{47}$$

with the coefficients

$$
\begin{gathered}
d_6 = |A|,\; d_5 = 0,\; d_4 = \mathbf{g}'A\mathbf{g} + \sum_{ij}^{3} C_{ij}a_{ij} \\
d_3 = \mathbf{g}'A\mathbf{c},\; d_2 = \mathbf{g}'C\mathbf{g} + \sum_{ij}^{3} c_{ij}A_{ij} \\
d_1 = \mathbf{g}'C\mathbf{c},\; d_0 = |C|
\end{gathered}
\tag{48}
$$

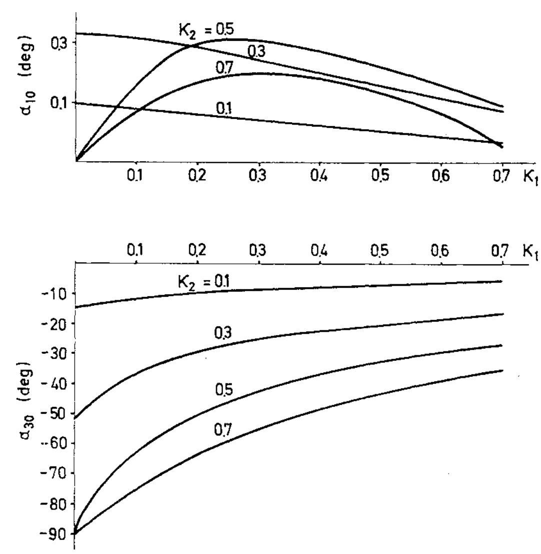

Fig. 5 Stable equilibrium orientations: example 2, case III.

where a_{ij} and c_{ij} are the cofactors of A_{ij} and C_{ij}, respectively. $\mathbf{g}$ is a vector formed by the elements of the skewsymmetric matrix G

$$\mathbf{g} = \begin{bmatrix} G_{23} \\ G_{31} \\ G_{12} \end{bmatrix} \tag{49}$$

and $\mathbf{c}$ is a vector formed by the skewsymmetric part of C

$$\mathbf{c} = \begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} = \begin{bmatrix} -(\partial M_{Aw_2}/\partial\alpha_3) + (\partial M_{Ay_3}/\partial\alpha_2) \\ \partial M_{Ax_1}/\partial\alpha_3 \\ -\partial M_{Ax_1}/\partial\alpha_2 \end{bmatrix} \tag{50}$$

The sum of the roots of the characteristic equation (47) is

$$\Sigma\lambda_i = -d_5/d_6 \tag{51}$$

Because d_5 is zero, at least one of the roots must have a positive real part if there is at least one root with a nonvanishing real part. Consequently, the equilibrium orientation E is unstable unless all roots are imaginary.

E is infinitesimally stable if all roots are imaginary. Infinitesimal stability, however, is a very weak statement because the higher-order terms, which have been neglected in Eq. (42), decide whether E is stable. Necessary and sufficient conditions for the roots to be imaginary can be given without calculating the roots explicitly. A necessary condition is

$$d_1 = d_3 = 0 \tag{52}$$

Because $\mathbf{g}$ is always unequal to zero, condition (52) can be satisfied only in the following two cases.

Case I:

$$\mathbf{c} = 0 \tag{53}$$

The matrix C has no skewsymmetric part and the conditions for conservative aerodynamic torque are satisfied at E. In this case the stability of E can be determined by checking the conditions (21) after the substitutions

$$U_{A22} \to -\partial M_{Aw_2}/\partial\alpha_2,\ U_{A23} \to -\partial M_{Aw_2}/\partial\alpha_3$$
$$U_{A33} \to -\partial M_{Ay_3}/\partial\alpha_3$$

are made.

Condition (53) is for instance satisfied if at E the 0_x system is coincident with the 0_y system. Then the center of pressure is on the y_1 axis and there results

$$(\partial M_{Aw_2}/\partial\alpha_3)|_{\text{at }E} = (\partial M_{Ay_3}/\partial\alpha_2)|_{\text{at }E} = 0$$
$$(\partial M_{Ax_1}/\partial\alpha_2)|_{\text{at }E} = (\partial M_{Ax_1}/\partial\alpha_3)|_{\text{at }E} = 0$$

Case II: Condition (52) is also satisfied for $\mathbf{c} \neq 0$ if $\mathbf{c}$ is perpendicular to both $\mathbf{g}'A$ and $\mathbf{g}'C$

$$\mathbf{c} \perp \mathbf{g}'C,\ \mathbf{c} \perp \mathbf{g}'A \tag{54}$$

If this condition holds, we can rewrite the characteristic equation (47) introducing $s = \lambda^2$

$$d_6 s^3 + d_4 s^2 + d_2 s + d_0 = 0 \tag{55}$$

To get imaginary roots for Eq. (47), the roots of Eq. (55) must be negative real. Consequently, the following conditions must hold:

$$d_6, d_4, d_2, d_0 > 0 \tag{56}$$

$$d_4 d_2 - d_6 d_0 > 0 \tag{57}$$

$$q^2 + p^3 < 0 \tag{58}$$

with

$$q = (d_4^3/27d_6^3) - (d_4 d_2/6d_6^2) + (d_0/2d_6)$$
$$p = (3d_6 d_2 - d_4^2)/9d_6^2$$

Conditions (56) and (57) are the Hurwitz relationships for the roots of Eq. (55) to have negative real parts, whereas Eq. (58) is the condition for the roots to be real.[8] The equilibrium orientation is unstable if at least one of the conditions (54 and 56–58) is violated. Especially condition (54) is a very strong restriction since it requires vector $\mathbf{c}$ to have a prescribed direction in three-dimensional space. Because there is only a very small chance that $\mathbf{c}$ will have exactly that direction, the equilibrium will in general be unstable for $\mathbf{c} \neq 0$, i.e., in case of nonconservative aerodynamic torque.

If the linearized equations of motion (42) for two of the δ_i's are uncoupled from the third one, the condition $\mathbf{c} = 0$ is necessary for stability.

IV. Summary of Analysis

The attitude stability of satellites influenced by gravity-gradient and aerodynamic torques has been investigated under the assumptions of circular orbits and constant atmospheric density.

In case of conservative aerodynamic torque, at least one stable equilibrium orientation exists. The stable orientations can be found either by solving Eqs. (19) and checking conditions (21), or by numerically determining the minimum of the dynamic potential, Eq. (18). The shape of the satellite and the location of the center of mass determine whether the aerodynamic torque is conservative. For all bodies of revolution with the center of mass located on the axis of symmetry, the aerodynamic torque is conservative.

Stable equilibrium orientations may also exist if the aerodynamic torque is conservative only in the vicinity of the equilibrium.

Nonconservative aerodynamic torques will in general cause a destabilization of all equilibrium orientations.

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